Welcome Back Puzzles 2018-19

Puzzle Contest 1 Solutions

(Parents and Grandparents)

1. **An evil dragon is tied to a pole in the ground** by an infinitely stretchy elastic cord attached to its tail. An armed grasshopper is on the pole watching the dragon. The dragon wants to run away. He jumps into the air and lands 10 m from the pole (with its tail still attached to the pole by the elastic cord). The grasshopper gives chase and leaps into the air, landing on the stretched elastic cord one meter from the pole. The dragon, seeing this, again leaps into the air and lands 10 m farther away from the pole (i.e., a total of 20 m from the pole).. The grasshopper bravely leaps into the air again, landing on the elastic cord 1 m further from where it was just sitting on the cord. Once again the dragon jumps another 10 m and then the grasshopper jumps another 1 m along the cord, leaving himself on the cord again. If this continues indefinitely, will the grasshopper ever catch up to the dragon? If it will, at what jump it will happen? ***(Based on a problem by Richard Feynman)*** **(60 pts.)**

Answer: It will happen at $(e^{10})^{th}$ jump

Solution: First note that the jumps of the dragon and the grasshopper alternate i.e. each of the grasshopper's jumps follows the dragon's jump. Let $i-1$ be the number of the jumps the dragon is done, $x(i-1)$ the position of the grosscooper at this moment. Then after the dragon has jumped i^{th} time the following happens:

1. The length of the cord is $10i\ m$. This is the dragon's position also.
2. Since an elastic cord any point of the cord gets displacement according to the infinitely stretchy elastic cord. Then the position of the grasshopper becomes $x(i-1) + x(i-1) \times \frac{10}{10(i-1)} = x(i-1)\left(1+\frac{1}{i-1}\right) = x(i-1) \times \frac{i}{i-1}$.
3. In addition the grasshopper makes its own i^{th} jump. Then $x(i) = x(i-1) \times \frac{i}{i-1} + 1$. This relationship is true for $i > 1$. Initially $x(1) = 1m$. Hence note

$$x(1) = 1$$

$$x(2) = x(1) \times \frac{2}{1} + 1 = 2 \times 1 \times \left(1 + \frac{1}{2}\right)$$
$$x(3) = x(2) \times \frac{3}{2} + 1 = 3 \times 1 \times \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$
$$\dots \dots \dots$$
$$x(n) = x(n-1) \times \frac{n}{n-1} + 1 = n \times 1 \times \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

The grasshopper can reach the dragon if

$$n \sum_{i=1}^{n} \frac{1}{i} = 10n \rightarrow \sum_{i=1}^{n} \frac{1}{i} = 10.$$

As known according to Euler

$$\sum_{i=1}^{n} \frac{1}{i} \approx \ln n$$

i.e. $n \approx e^{10} \approx 22026$

2. At the end of each summer, the fairy of flowers

arranges a colorful performance on her flower bed, which has the shape of a circle divided into seven equal parts. Each part, planted with beautiful flowers, the fairy colors in one of the colors of the rainbow, creating a unique arrangement of these seven colors every second. How long does this color play last? (In an arrangement, two or more parts may have the same color. However, two arrangements of seven colors are the same if they coincide when turning a flower bed) **(40 pts.)**

Answer: 117655 s or $\approx 32\ h\ 40\ min$

Solution: Note the following:

1. Since each of the seven parts of the flower bed can be colored in one of the seven rainbow colors, there are 7^7 possible arrangements
2. Among them there are 7 arrangements that are colored in one color for each
3. The remaining $7^7 - 7$ arrangements can be divided into 7 classes. In addition, any two arrangements belonging to the same class coincide because of the circular symmetry. It means that there are $\frac{7^7-7}{7} = 7^6 - 1$ unique arrangements
4. Then there are $7^6 - 1 + 7 = 7^6 + 6 = 117,655$ unique colorings of the flower bed
5. It takes $117,655$ sec to show all of them.