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2017-2018 Puzzle Contests

Solutions for Contest #4

Parents and Grandparents Puzzle Solutions:

1. A black Rook and the white King are arranged on an 8 x 8 chess board so that the Rook threatens the King but the King does not threaten the Rook. How many such positions of the King and the Rook are possible? (The positions are identical if they can be obtained from each other by a rotation of the board) **(25 pts.)**

Answer: 672

Solution: Note the following:

1. For any position of a Rook it holds 14 chessboard squares under threat that are 7 horizontal and 7 vertical squares.
2. If the King's position is in a corner of the board then we have to exclude one horizontal and one vertical square nearest to the King. It means that in this case we have 4 x 12 = 48 possible positions for the Rook and the King
3. If the King's position is at an edge of the board but not in the corner, (there are 24 such swuares) then we have to exclude one horizontal and two vertical squares or two horizontal and one vertical square that are nearest to the King. It means that in this case we have 24 x 11 = 264 possible positions for the Rook and the King
4. If the King's position is not at an edge of the board (there are 6 x 6 = 36 such squares) then we have to exclude two horizontal and two vertical squares that are nearest to the King. It means that in this case we have 36 x 10 = 360 possible positions for the Rook and the King
5. Finally we have 48 + 264 + 360 = 672 possible positions

2. On the Island of Eternal Love the alphabet consists of just two letters  and  only. In addition no word in its language has two consecutive letters , and no word is longer than 8 letters. How many possible words are there in the language of the Island of Eternal Love? **(35 pts.)**

Answer: 141 words

Solution: Let us use x for the letter  and y for the letter . Denote a_n the number of words of the length n (n-words) in the language and note that any word of the language starts whether x or y. Then it is easy to note that $a_1 = 2: (x, y); a_2 = 3: (xx, xy, yx)$. Note also that

- if an n-word starts from x then the next $n-1$ letters have to form a_{n-1} words,
- if an n-word starts from y then the next letter has to be x and $n-2$ letters have to form a_{n-2} words ,

i.e.
$a_n = a_{n-1} + a_{n-2}$. It is known as the Fibonacci sequence for which $a_1 = 2, a_2 = 3, a_3 = 2 + 3 = 5, a_4 = 3 + 5 = 8, a_5 = 5 + 8 = 13, a_6 = 8 + 13 = 21, a_7 = 13 + 21 = 34, a_8 = 21 + 34 = 55$. Then there are $2 + 3 + 5 + 8 + \cdots + 55 = 141$ words in total.

3. On Valentine's Day, citizens of the Island of Eternal Love like to play the following word game. The two letters  and  can be placed in the cells of a 100 x 100 table. A player is allowed to change any letter in any cell, but if the letter is changed, then all the letters in the corresponging column **and** row must also change its state on that move. If all cells in the table are initially filled with letter , it possible to come up with a sequence of moves that will result a table with exactly 2018 letters  after some number of moves? Why or why not? **(40 pts.)**

Answer: Yes, it is possible.
Solution: **(this elegant solution is suggested by Jason Liao, father of Caden Liao)**
Note the following:

1. For any even natural $2 \leq n \leq 100$ flipping once all cells of $n \times n$ square changes all signs in the square but doesn't change any sign which is out of the $n \times n$ square. It is true because any cell in that square gets flipped $2 \cdot n - 1$ times, which is odd and the outer cells are flipped n times which is even
2. $2018 = 394 + 40^2 + 4^2 + 2^2 + 2^2$
3. Let us index the cell of the 100×100 table as $(1, 1), (1, 2), \dots, (100, 100)$.
4. Flip once the cells $(99, 99)$ and $(100, 100)$ in any order. We get $2 \times 100 - 1 + 2 \times 99 - 1 - 2 = 394$ cells that changed signs; after that choose without overlapping the $40 \times 40, 4 \times 4$ and two 2×2 squares and flip once each cell of these squares. Then according to 1., 2. we have $394 + 1600 + 16 + 4 + 4 = 2018$ the changed signs